

# Semi-orthogonal Parseval wavelets associated to GMRA on local fields of positive characteristics

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## Abstract

In this article we establish theory of semi-orthogonal Parseval wavelets associated to generalized multiresolution analysis (GMRA) for the local field of positive characteristics (LFPC). By employing the properties of translation invariant spaces on the core space of GMRA we obtain a characterization of semi-orthogonal Parseval wavelets in terms of consistency equation for LFPC. As a consequence, we obtain a characterization of an orthonormal (multi)wavelet to be associated with an MRA in terms of multiplicity function as well as dimension function of a (multi)wavelet. Further, we provide characterizations of Parseval scaling functions, scaling sets and bandlimited wavelets together with a Shannon type multiwavelet for LFPC.

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## 1. Introduction

In the recent years wavelets on local fields of positive characteristics (LFPC) are extensively studied by many authors including Behera and Jahan, Benedetto, Debnath and Shah, Jiang, Li and Ji, Vyas and first author with respect to multiresolution analysis (MRA), frame multiresolution analysis, tight wavelet frame, low-pass filter, etc. in the references [4-6, 18, 20, 22-24] but still more concepts need to be studied for its enhancement. Indeed, the development of theory of wavelet analysis with respect to groups other than Euclidean spaces, namely,  $p$ -adic groups, Cantor dyadic groups, Vilenkin groups, locally compact abelian groups (LCAG), Heisenberg group, etc., was always an interesting part for researchers due to its various applications [1, 6, 13, 16, 17, 19].

Our main goal is to develop the theory of semi-orthogonal Parseval (multi)wavelets associated to generalized multiresolution analysis (GMRA) in the setting of LFPC while a rigorous study of semi-orthogonal Parseval (multi)wavelets and GMRA has been done by many authors for the case of Euclidean spaces [2, 3, 8-10]. The concept of GMRA was introduced by Baggett, Medina and Merrill in [2] for separable Hilbert spaces. They used unitary representation of the group of translations acting on the fundamental subspace  $V_0$  (known as, *core space*) that generates, through dilation, the subspaces  $\{V_j\}_{j \in \mathbb{Z}}$  associated with the GMRA.

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As the core space  $V_0$  possess the properties of translation invariant (TI) spaces and the concept of TI spaces plays a very important role in the development of theory of GMRA's for the case of Euclidean spaces [3, 7, 9, 10], we adopt the approach of TI spaces to establish the theory of semi-orthogonal Parseval multiwavelets associated with GMRA's in the setting of LFPC and obtain necessary and sufficient conditions for the existence of a semi-orthogonal Parseval (multi)wavelet in terms of consistency equation. Further, we obtain a characterization of an orthonormal (multi)wavelet to be associated with an MRA in terms of dimension function of a (multi)wavelet in the setting of LFPC. For this, we provide a brief introduction of TI spaces for the case of LFPC while Currey, Mayeli and Oussa in [14] for nilpotent Lie groups, and Bownik and Ross in [11] for LCAG having a co-compact subgroup developed the theory of TI spaces. Also Bownik [7] and Rzeszutnik [21] studied the same in the Euclidean setting.

Although minimally supported frequency wavelets (wavelet sets) are not well-localized, and thus not directly useful for applications, they have proved to be an essential tool in developing wavelet theory. The existence of wavelet sets for LCAG and related groups, and Heisenberg groups have been discussed by Benedetto and Benedetto in [6], and Currey and Mayeli in [13], respectively. Wavelet sets are also studied in the references [2, 3, 8, 15, 21]. We also characterize bandlimited Parseval multiwavelet sets of finite order which in turn characterizes all multiwavelet sets for LFPC. Further, we provide a necessary and sufficient condition of scaling functions associated with Parseval multiwavelets which provides characterization of scaling sets. In this setting, we obtain a Shannon type multiwavelet associated to MRA.

The present paper is organized as follows:

In Section 2, we provide brief introduction about the local field of positive characteristics. For this we refer a book by Taibleson [25]. Section 3, is divided into three subsections. In the first subsection, we discuss about core spaces of semi-orthogonal framelets while second subsection contains a characterization of translation invariant spaces along with some basic notions such as range function, multiplicity function and spectral function for LFPC. In the last subsection, we discuss semi-orthogonal Parseval multiwavelets, GMRA's and their main characterization theorem that provide a connection between the dimension function and the multiplicity function of a (multi)wavelet. We also prove that the wavelet multiplicity function satisfies a consistency equation, and the multiplicity function is equal to one in case of multiwavelets associated to MRA's. Finally, in Section 4, we provide a characterization of bandlimited Parseval frame multiwavelets for LFPC, and necessary and sufficient conditions of Parseval scaling functions which generalizes the characterization of orthonormal scaling functions provided by Behera and Jahan [4].

## 2. Basic Results and notations of LFPC

Throughout the paper,  $K$  denotes a local field. By a local field we mean a field which is locally compact, non-discrete, and totally disconnected. The set  $\mathcal{O} = \{x \in K : |x| \leq 1\}$  denotes the ring of integers which is a unique maximal compact open subring of  $K$ , where the absolute value  $|x|$  of  $x \in K$  satisfies the properties:  $|x| = 0$  if and only if  $x = 0$ ;  $|xy| = |x||y|$ , and  $|x + y| \leq \max\{|x|, |y|\}$ , for all  $x, y \in K$ . Define  $\mathfrak{P} = \{x \in K : |x| < 1\}$ , which is called the *prime ideal* in  $K$ . In view of totally disconnectedness of  $K$ , there exists an element  $\mathfrak{p}$  (known as *prime element*) of  $\mathfrak{P}$  having maximum absolute value and then  $\mathfrak{P} = \mathfrak{p}\mathcal{O}$ . It can be easily proved that,  $\mathfrak{P}$  is compact and open. Therefore, the residue space  $\mathcal{Q} = \mathcal{O}/\mathfrak{P}$  is isomorphic to a finite field  $GF(q)$ , where  $q = p^c$  for some prime  $p$  and positive integer  $c$ .

For a measurable subset  $E$  of  $K$ , let  $|E| = \int_K \chi_E(x) dx$ , where  $\chi_E$  is the characteristic function of  $E$  and  $dx$  is the Haar measure for  $K^+$  (locally compact additive group of  $K$ ), so  $|\mathcal{O}| = 1$ . By decomposing  $\mathcal{O}$  into  $q$  cosets of  $\mathfrak{P}$ , we have  $|\mathfrak{P}| = q^{-1}$  and  $|\mathfrak{p}| = q^{-1}$ , and hence for  $x \in K \setminus \{0\} =: K^*$  (locally compact multiplicative group of  $K$ ), we have  $|x| = q^k$ , for some  $k \in \mathbb{Z}$ . Further, notice that  $\mathcal{O}^* := \mathcal{O} \setminus \mathfrak{P}$  is the group of units in  $K^*$ , and for  $x \neq 0$ , we may write  $x = \mathfrak{p}^k x'$  with  $x' \in \mathcal{O}^*$ . In the sequel, we denote  $\mathfrak{p}^k \mathcal{O}$  by  $\mathfrak{P}^k$ , for each  $k \in \mathbb{Z}$  that is known as *fractional ideal*. Here, for  $x \in \mathfrak{P}^k$ ,  $x$  can be expressed uniquely as  $x = \sum_{l=k}^{\infty} c_l \mathfrak{p}^l$ ,  $c_l \in \mathfrak{U}$ , and  $c_k \neq 0$ , where  $\mathfrak{U} = \{c_i\}_{i=0}^{q-1}$  is a fixed full set of coset representatives of  $\mathfrak{P}$  in  $\mathcal{O}$ .

Let  $\chi$  be a fixed character on  $K^+$  that is trivial on  $\mathcal{O}$  but is nontrivial on  $\mathfrak{P}^{-1}$ , which can be found by starting with nontrivial character and rescaling. For  $y \in K$ , we define  $\chi_y(x) = \chi(yx)$ ,  $x \in K$ . For  $f \in L^1(K)$ , the *Fourier transform* of  $f$  is the function  $\hat{f}$  defined by

$$\hat{f}(\xi) = \int_K f(x) \overline{\chi_\xi(x)} dx = \int_K f(x) \chi(-\xi x) dx,$$

which can be extended for  $L^2(K)$ .

Notation  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . Let  $\chi_u$  be any character on  $K^+$ . Since  $\mathcal{O}$  is a subgroup of  $K^+$ , it follows that the restriction  $\chi_u|_{\mathcal{O}}$  is a character on  $\mathcal{O}$ . Since  $\chi_u|_{\mathcal{O}}$  is a character on  $\mathcal{O}$ , we have  $\chi_u = \chi_v$  if and only if  $u - v \in \mathcal{O}$ . Hence, we have the following result [25, Proposition 6.1]:

**Theorem 2.1.** *Let  $\mathcal{Z} := \{u(n)\}_{n \in \mathbb{N}_0}$  be a complete list of (distinct) coset representation of  $\mathcal{O}$  in  $K^+$ . Then  $\{\chi_{u(n)}|_{\mathcal{O}} \equiv \chi_{u(n)}\}_{n \in \mathbb{N}_0}$  is a list of (distinct) characters on  $\mathcal{O}$ . Moreover, it is a complete orthonormal system on  $\mathcal{O}$ .*

Next, we proceed to impose a natural order on  $\mathcal{Z}$  which is used to develop the theory of Fourier series on  $L^2(\mathcal{O})$ . For this, we choose a set  $\{1 = \epsilon_0, \epsilon_i\}_{i=1}^{c-1} \subset \mathcal{O}^*$  such that the vector space  $\mathcal{Q}$  generated by  $\{1 = \epsilon_0, \epsilon_i\}_{i=1}^{c-1}$  is isomorphic to the vector space  $GF(q)$  over finite field  $GF(p)$  of order  $p$  as  $q = p^c$ . For  $n \in \mathbb{N}_0$  such that  $0 \leq n < q$ , we write  $n = \sum_{k=0}^{c-1} a_k p^k$ , where  $0 \leq a_k < p$ . By noting that  $\{u(n)\}_{n=0}^{q-1}$ , a complete set of coset representatives of  $\mathcal{O}$  in

$\mathfrak{P}^{-1}$  with  $|u(n)| = q$ , for  $0 < n < q$  and  $u(0) = 0$ , we define  $u(n) = (\sum_{k=0}^{c-1} a_k \epsilon_k) \mathfrak{p}^{-1}$ . Now, for  $n \geq 0$ , we write  $n = \sum_{k=0}^s b_k q^k$ , where  $0 \leq b_k < q$ , and define  $u(n) = \sum_{k=0}^s u(b_k) \mathfrak{p}^{-k}$ . In general, it is not true that  $u(m+n) = u(m) + u(n)$  but  $u(rq^k + s) = u(r) \mathfrak{p}^{-k} + u(s)$ , if  $r \geq 0$ ,  $k \geq 0$  and  $0 \leq s < q^k$ .

Now, we sum up above in the following theorem (see, [25, Proposition 6.6], [4]):

**Theorem 2.2.** *For  $n \in \mathbb{N}_0$ , let  $u(n)$  be defined as above.*

- (a)  $u(n) = 0$  if and only if  $n = 0$ . If  $k \geq 1$ , then we have  $|u(n)| = q^k$  if and only if  $q^{k-1} \leq n < q^k$ .
- (b)  $\{u(k) : k \in \mathbb{N}_0\} = \{-u(k) : k \in \mathbb{N}_0\}$ .
- (c) For a fixed  $l \in \mathbb{N}_0$ , we have  $\{u(l) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ .

Following result and definition [18] will be used in the sequel:

**Theorem 2.3.** *For all  $l, k \in \mathbb{N}_0$ ,  $\chi_{u(k)}(u(l)) = 1$ .*

**Definition 2.4.** A function  $f$  defined on  $K$  is said to be *integral periodic* if

$$f(x + u(l)) = f(x), \quad \text{for all } l \in \mathbb{N}_0, x \in K.$$

### 3. Multiplicity function associated to Semi-orthogonal Parseval wavelets

We begin with Subsection 3.1 by constructing some examples of Parseval multiwavelets along with Shannon type multiwavelet for LFPC and also, core spaces associated with semi-orthogonal framelets. Then we provide a brief introduction about translation invariant spaces in Subsection 3.2. Finally, we find a connection between semi-orthogonal Parseval multiwavelets and generalized multiresolution analysis by providing a consistency equation in terms of multiplicity function of core space in Subsection 3.3, which is the main aim of this section. As a consequence, we obtain an expected result that the multiplicity function associated to an MRA multiwavelets is equal to 1.

#### 3.1 Core spaces of Semi-orthogonal framelets

In order to define the concepts of MRA and wavelets on LFPC  $K$ , we need notions of translation and dilation. Since  $\bigcup_{j \in \mathbb{Z}} \mathfrak{p}^{-j} \mathcal{O} = K$ , we can regard  $\mathfrak{p}^{-1}$  as the dilation (note that  $|\mathfrak{p}^{-1}| = q$ ) and since  $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$  is a complete list of distinct coset representatives of  $\mathcal{O}$  in  $K$ , the set  $\mathcal{Z}$  can be treated as the translation set. Note that it follows from Theorem 2.2 that the translation set  $\mathcal{Z}$  is a subgroup of  $K^+$  even though it is indexed by  $\mathbb{N}_0$ . We have the following definition:

**Definition 3.1.1.** A finite set  $\Psi = \{\psi_m : m = 1, 2, \dots, L\} \subset L^2(K)$  is called a *frame wavelet* (simply, *framelet*) in  $L^2(K)$  if the system

$$\mathcal{A}(\Psi) := \left\{ D_{\mathfrak{p}}^j T_k \psi_m : 1 \leq m \leq L, j \in \mathbb{Z}, k \in \mathbb{N}_0 \right\}$$

forms a frame for  $L^2(K)$ , that means, for each  $f \in L^2(K)$ , there are  $0 < A \leq B < \infty$  such that

$$A\|f\|^2 \leq \sum_{m=1}^L \sum_{(j,k) \in \mathbb{Z} \times \mathbb{N}_0} \left| \langle f, D_{\mathfrak{p}}^j T_k \psi_m \rangle \right|^2 \leq B\|f\|^2,$$

where the dilation and translation operators are defined as follows:

$$D_{\mathfrak{p}}^j f(x) = q^{j/2} f(\mathfrak{p}^{-j} x), \text{ and } T_k f(x) = f(x - u(k)), \text{ for } x \in K, j \in \mathbb{Z}, k \in \mathbb{N}_0.$$

For  $A = B$ ,  $\Psi$  is known as *tight frame wavelet* (simply, *tight framelet*) with constant  $A$  while it is known as *Parseval multiwavelet* of order  $L$  for  $A = B = 1$ . In case of Parseval frame system  $\mathcal{A}(\{\psi\})$  for  $L^2(K)$ ,  $\psi$  is known as *Parseval wavelet*. If the system  $\mathcal{A}(\Psi)$  is an orthonormal basis for  $L^2(K)$ ,  $\Psi$  is called an *orthonormal multiwavelet* (simply, *multiwavelet*) of order  $L$  of  $L^2(K)$ . Moreover, a framelet  $\Psi$  is known as *semi-orthogonal* if  $D_{\mathfrak{p}}^j W \perp D_{\mathfrak{p}}^{j'} W$ , for  $j \neq j'$ , where  $W = \overline{\text{span}}\{T_k \psi : k \in \mathbb{N}_0, \psi \in \Psi\}$ .

For  $f \in L^2(K)$  and  $j, k \geq 0$ , we have

$$\begin{aligned} T_k D_{\mathfrak{p}}^j f(x) &= D_{\mathfrak{p}}^j f(x - u(k)) = q^{j/2} f(\mathfrak{p}^{-j} x - \mathfrak{p}^{-j} u(k)) \\ &= q^{j/2} f(\mathfrak{p}^{-j} x - u(q^j k)) = q^{j/2} T_{q^j k} f(\mathfrak{p}^{-j} x) \\ &= D_{\mathfrak{p}}^j T_{q^j k} f(x), \end{aligned}$$

which shows that  $T_k D_{\mathfrak{p}}^j = D_{\mathfrak{p}}^j T_{q^j k}$ , for all  $j, k \geq 0$ . Further, we notice that for  $f \in L^2(K)$  and  $\xi \in K$ , we have

$$\left( \widehat{D_{\mathfrak{p}}^j T_k f} \right)(\xi) = q^{-j/2} \chi_{u(k)}(-\mathfrak{p}^j \xi) \widehat{f}(\mathfrak{p}^j \xi), \text{ for } j \in \mathbb{Z}, k \in \mathbb{N}_0.$$

The following is a necessary and sufficient condition for the system  $\mathcal{A}(\Psi)$  to be a Parseval frame for  $L^2(K)$  [5].

**Theorem 3.1.2.** Suppose  $\Psi = \{\psi_m : m = 1, 2, \dots, L\} \subset L^2(K)$ . Then the affine system  $\mathcal{A}(\Psi)$  is a Parseval frame for  $L^2(K)$  if and only if for a.e.  $\xi$ , the following holds:

- (i)  $\sum_{m=1}^L \sum_{j \in \mathbb{Z}} \left| \widehat{\psi}_m(\mathfrak{p}^{-j} \xi) \right|^2 = 1,$
- (ii)  $\sum_{m=1}^L \sum_{j \in \mathbb{N}_0} \widehat{\psi}_m(\mathfrak{p}^{-j} \xi) \overline{\widehat{\psi}_m(\mathfrak{p}^{-j}(\xi + u(s)))} = 0, \text{ for } s \in \mathbb{N}_0 \setminus q\mathbb{N}_0.$

In particular,  $\Psi$  is an orthonormal multiwavelet of order  $L$  in  $L^2(K)$  if and only if  $\|\psi_m\| = 1$ , for  $1 \leq m \leq L$ , and the above conditions (i) and (ii) hold.

**Definition 3.1.3.** A *multiresolution analysis* (MRA) of  $L^2(K)$  is a sequence of closed subspaces  $\{D_{\mathfrak{p}}^j(V)\}_{j \in \mathbb{Z}}$  of  $L^2(K)$  satisfying the following properties:

- (M1)  $T_k V = V$ , (M2)  $V \subset D_{\mathfrak{p}}(V)$ , (M3)  $\bigcap_{j \in \mathbb{Z}} D_{\mathfrak{p}}^j(V) = \{0\}$ , (M4)  $\overline{\bigcup_{j \in \mathbb{Z}} D_{\mathfrak{p}}^j(V)} = L^2(K)$ ,  
(M5) there is a  $\varphi \in V$  (known as *orthonormal scaling function*) such that  $\{T_k \varphi\}_{k \in \mathbb{N}_0}$  forms an orthonormal basis for  $V$ .

The space  $V$  is known as *core space*. If we replace (M5) by (M6) as follows:

- (M6) the system  $\{T_k \varphi\}_{k \in \mathbb{N}_0}$  forms a Parseval frame for  $V$ ,  
then the sequence  $\{D_{\mathfrak{p}}^j(V)\}_{j \in \mathbb{Z}}$  is known as *Parseval multiresolution analysis* (PMRA), and  $\varphi$  is known as *Parseval scaling function*.

Next, we provide an example of orthonormal multiwavelet of order  $q - 1$  which is of *Shannon type* for  $L^2(K)$ :

**Example 3.1.4.** Let us consider the ring of integers  $\mathcal{O}$  in  $K$ , and let the set  $\{u(n)\}_{n=0}^{q-1}$  be a complete set of distinct coset representatives of  $\mathcal{O}$  in  $\mathfrak{P}^{-1}$  with  $u(0) = 0$ , and  $|u(n)| = q$ , for  $0 < n < q$ . Suppose  $\Psi = \{\psi_1, \psi_2, \dots, \psi_{q-1}\}$ , is a collection of functions in  $L^2(K)$  such that the Fourier transform

$$\widehat{\psi}_i(\xi) = \chi_{\mathcal{O}+u(i)}(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathcal{O} + u(i) \\ 0 & \text{otherwise,} \end{cases}$$

for all  $i, 1 \leq i \leq q - 1$ . Now, we show that  $\Psi$  satisfies all the conditions of Theorem 3.1.2.

(a) For each  $i, 1 \leq i \leq q - 1$ ,  $\|\psi_i\|^2 = \|\widehat{\psi}_i\|^2 = |\mathcal{O} + u(i)| = |\mathcal{O}| = 1$ .

(b) Since  $\mathcal{O}$  is an additive subgroup of  $\mathfrak{P}^{-1}$ , we have

$$\{\mathcal{O} + u(0), \mathcal{O} + u(1), \dots, \mathcal{O} + u(q - 1)\},$$

a measurable partition of  $\mathfrak{P}^{-1}$ , and hence the system  $\{\mathcal{O} + u(1), \dots, \mathcal{O} + u(q - 1)\}$  is a measurable partition of the set  $\mathfrak{P}^{-1} \setminus \mathcal{O} = \mathfrak{p}^{-1} \mathcal{O}^*$  as  $u(0) = 0$ . As  $\bigcup_{j \in \mathbb{Z}} \mathfrak{p}^{-j} \mathcal{O} = K$ , and

$\mathcal{O} \subset \mathfrak{p}^{-1} \mathcal{O} = \mathfrak{P}^{-1}$ , we have a measurable partition  $\{\mathfrak{p}^{-j} \mathcal{O}^* : j \in \mathbb{Z}\}$  of  $K$  and hence  $\{\mathfrak{p}^{-j}(\mathcal{O} + u(i)) : j \in \mathbb{Z}, 1 \leq i \leq q - 1\}$  is a measurable partition of  $K$ . This shows that

$$\sum_{i=1}^{q-1} \sum_{j \in \mathbb{Z}} \left| \widehat{\psi}_i(\mathfrak{p}^{-j} \xi) \right|^2 = 1, \text{ for a.e. } \xi.$$

- (c) The term  $\widehat{\psi}_i(\mathbf{p}^{-j}\xi)\overline{\widehat{\psi}_i(\mathbf{p}^{-j}(\xi + u(s)))} = \widehat{\psi}_i(\mathbf{p}^{-j}\xi)\overline{\widehat{\psi}_i(\mathbf{p}^{-j}\xi + u(q^j s))}$  is nonzero only when  $\mathbf{p}^{-j}\xi$  and  $\mathbf{p}^{-j}\xi + u(q^j s)$  are in the support of  $\widehat{\psi}_i$ , denoted by  $\text{supp } \widehat{\psi}_i$ , for  $1 \leq i \leq q-1$ . But, for  $s \in \mathbb{N}_0 \setminus q\mathbb{N}_0$  and  $1 \leq i \leq q-1$ , both  $\mathbf{p}^{-j}\xi$  and  $\mathbf{p}^{-j}\xi + u(q^j s)$  will not be members of  $\text{supp } \widehat{\psi}_i = \mathcal{O} + u(i)$  since  $\{\mathcal{O} + u(k) : k \in \mathbb{N}_0\}$  is a measurable partition of  $K$ . Hence we have

$$\sum_{i=1}^{q-1} \sum_{j \in \mathbb{N}_0} \widehat{\psi}_i(\mathbf{p}^{-j}\xi)\overline{\widehat{\psi}_i(\mathbf{p}^{-j}(\xi + u(s)))} = 0, \quad \text{for } s \in \mathbb{N}_0 \setminus q\mathbb{N}_0.$$

This proves that,  $\Psi$  is an orthonormal multiwavelet of order  $q-1$ .

Following are examples of Parseval multiwavelets of order 1 as well as  $q-1$  in  $L^2(K)$ :

**Example 3.1.5. (a).** Let  $m \in \mathbb{N}$ . Suppose  $\psi$  is a function in  $L^2(K)$  whose Fourier transform is defined as follows:

$$\widehat{\psi}(\xi) = \chi_{\mathbf{p}^m \mathcal{O}^*}(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbf{p}^m \mathcal{O}^* = \mathfrak{P}^m \setminus \mathfrak{P}^{m+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then,  $\psi$  is a Parseval multiwavelet of order 1 in view of the following:

- (i) The system  $\{\mathbf{p}^j(\mathbf{p}^m \mathcal{O}^*) : j \in \mathbb{Z}\}$  is a measurable partition of  $K$  since  $\bigcup_{j \in \mathbb{Z}} \mathbf{p}^{-j} \mathcal{O} = K$ ,  $\mathcal{O} \subset \mathfrak{P}^{-1}$ , and  $\mathbf{p}^{-1} \mathcal{O}^* = \mathfrak{P}^{-1} \setminus \mathcal{O}$ .
- (ii) The system  $\{\mathbf{p}^m \mathcal{O}^* + u(k) : k \in \mathbb{N}_0\}$  is a measurable partition of a measurable subset of  $K$  since  $\{\mathcal{O} + u(k) : k \in \mathbb{N}_0\}$  is a measurable partition of  $K$  and  $\mathfrak{P}^m \subset \mathcal{O}$ .

**(b).** Let  $m \in \mathbb{N}$ . Suppose  $\Psi = \{\psi_1, \psi_2, \dots, \psi_{q-1}\}$  is a collection of functions in  $L^2(K)$  whose Fourier transforms are defined as follows for each  $i, 1 \leq i \leq q-1$ :

$$\widehat{\psi}_i(\xi) = \chi_{\mathfrak{P}^m + \mathbf{p}^m u(i)}(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathfrak{P}^m + \mathbf{p}^m u(i) \\ 0 & \text{otherwise.} \end{cases}$$

Then, in view of Example 3.1.4, with  $\mathfrak{P}^m = \mathbf{p}^m \mathcal{O}$ ,  $\mathfrak{P}^m \subset \mathcal{O}$ , and the system  $\{\mathbf{p}^m W_i : 1 \leq i \leq q-1\}$  is a measurable partition of  $\mathbf{p}^{m-1} \mathcal{O}^*$ , where  $W_i = \mathcal{O} + u(i)$ ,  $\Psi$  is a Parseval multiwavelet of order  $q-1$ . Here, note that  $|\mathfrak{P}^m| = \frac{1}{q^m} < 1$ , as  $q \geq 2$ , and for  $k, k' \in \mathbb{N}_0$ , ( $k \neq k'$ ), we have

$$\begin{aligned} |(\mathbf{p}^m W_i + u(k)) \cap (\mathbf{p}^m W_i + u(k'))| &= q^m |(W_i + \mathbf{p}^{-m} u(k)) \cap (W_i + \mathbf{p}^{-m} u(k'))| \\ &= q^m |(W_i + u(q^m k)) \cap (W_i + u(q^m k'))| \\ &= 0, \end{aligned}$$

as the system  $\{W_i + u(k) : k \in \mathbb{N}_0\}$  is a measurable partition of  $K$ .

Now, we state a lemma which is a basic perturbation result for frames stated in [12] that will be useful to construct examples of framelets but not Parseval (multi)wavelets for LFPC with the help of above examples:

**Lemma 3.1.6.** *Suppose that  $H$  is a Hilbert space,  $\{f_j\} \subset H$  is a frame with constants  $C_1$  and  $C_2$ ,*

$$C_1 \|f\| \leq \sum_j |\langle f, f_j \rangle|^2 \leq C_2 \|f\| \quad \text{for all } f \in H,$$

*and  $\{g_j\} \subset H$  is a Bessel sequence with constant  $C_0$ ,*

$$\sum_j |\langle f, g_j \rangle|^2 \leq C_0 \|f\|_2^2 \quad \text{for all } f \in H.$$

*If  $C_0 < C_1$ , then  $\{f_j + g_j\}$  is a frame with constants  $((C_1)^{1/2} - (C_0)^{1/2})^2$  and  $((C_2)^{1/2} + (C_0)^{1/2})^2$ .*

In Example 3.1.4, the core space  $V$  of Shannon type multiwavelet is given by

$$V = \overline{\text{span}}\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0, |\widehat{\varphi}| = \chi_S\},$$

where  $S = \mathcal{O}$ . Notice that the sequence  $\{D_{\mathbf{p}}^j V\}_{j \in \mathbb{Z}}$  satisfies all the axioms of MRA. While the core spaces  $V$  and  $V'$  of part (a) and part (b) of Example 3.1.5 are defined as follows:

$$V = \overline{\text{span}}\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0, |\widehat{\varphi}| = \chi_{\mathfrak{p}^{m+1}}\}, \quad \text{and} \\ V' = \overline{\text{span}}\left\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0, |\widehat{\varphi}| = \chi_{\bigcup_{i=1}^{q-1} \mathfrak{p}^j \cup \bigcup_{j \in \mathbb{N}} \mathfrak{p}^j (\mathfrak{p}^m W_i)}\right\},$$

respectively in which  $\{D_{\mathbf{p}}^j V\}_{j \in \mathbb{Z}}$  and  $\{D_{\mathbf{p}}^j V'\}_{j \in \mathbb{Z}}$  satisfy all the axioms of MRA other than axiom (M5).

Such sequences  $\{D_{\mathbf{p}}^j V\}_{j \in \mathbb{Z}}$  satisfying conditions (M1) – (M4) are known as *generalized Multiresolution analyses* (GMRA), the concept of which was introduced by Baggett, Medina and Merrill in [2] for separable Hilbert spaces. They developed GMRA structure for  $L^2(\mathbb{R}^n)$ . Since the core space  $V$  possess the property of TI space, it motivates us to use the theory of TI spaces for  $L^2(K)$  for developing connection between the GMRA structure and (multi)wavelets or framelets for  $L^2(K)$ .

Given a finite family  $\Psi \subset L^2(K)$ , we define its *space of negative dilates*  $V$  as follows:

$$V = \overline{\text{span}}\{q^{j/2} \psi(\mathfrak{p}^j \cdot -u(k)) : j < 0, k \in \mathbb{N}_0, \psi \in \Psi\}.$$

We say that a framelet  $\Psi$  comes from a GMRA if its space of negative dilates  $V$  satisfies (M1) – (M4). In addition, if  $V$  satisfies (M5), then  $V$  is associated with an MRA.



Since for a semi-orthogonal framelet  $\Psi$ , its space of negative dilates  $V$  and the space  $W = \overline{\text{span}}\{T_k\psi : k \in \mathbb{N}_0, \psi \in \Psi\}$  satisfy

$$\bigoplus_{j \in \mathbb{Z}} D_{\mathbf{p}}^j W = L^2(K), \quad V = \bigoplus_{j \leq -1} D_{\mathbf{p}}^j W = \left( \bigoplus_{j \geq 0} D_{\mathbf{p}}^j W \right)^\perp,$$

it can be easily seen that every semi-orthogonal framelet  $\Psi$  comes from a GMRA.

A study of translation invariant spaces will be useful to see its converse, that is, when a GMRA gives rise to a (multi)wavelet, or a semi-orthogonal framelet.

### 3.2. Translation invariant spaces for LFPC

Suppose  $\mathcal{Z} = \{u(k) : k \in \mathbb{N}_0\}$ . A closed subspace  $V$  of  $L^2(K)$  is said to be *translation invariant* under  $\mathcal{Z}$ , in short  $\mathcal{Z}$ -TI if  $f \in V$  implies  $T_k f \in V$  for all  $k \in \mathbb{N}_0$ . Given a countable subset  $\mathcal{A} \subset L^2(K)$ , we define a  $\mathcal{Z}$ -TI space generated by  $\mathcal{A}$  as

$$\mathcal{S}^{\mathcal{Z}}(\mathcal{A}) = \overline{\text{span}}\{T_k f : f \in \mathcal{A}, k \in \mathbb{N}_0\} \subset L^2(K).$$

If  $\mathcal{A} = \{\varphi\}$ , then the space  $\mathcal{S}^{\mathcal{Z}}(\{\varphi\})$  is called *principal translation invariant* under  $\mathcal{Z}$  ( $\mathcal{Z}$ -PTI) which is denoted by  $V_\varphi$ .

It is easy to see in view of Zorn's lemma that any  $\mathcal{Z}$ -TI space can be decomposed into an orthogonal sum of  $\mathcal{Z}$ -PTI spaces. That means, if  $V$  is a  $\mathcal{Z}$ -TI space, then there exists a countable set of functions  $\{\varphi_i\}_{i=1}^M$  belonging to  $V$  such that  $V = \bigoplus_{i=1}^M V_{\varphi_i}$ , where  $M$  is a natural number or infinity. Here, the orthogonality condition of  $\mathcal{Z}$ -PTI spaces, namely,  $V_{\varphi_1}$  and  $V_{\varphi_2}$  can be written as :

$$\sum_{k \in \mathbb{N}_0} \widehat{\varphi}_1(\xi + u(k)) \overline{\widehat{\varphi}_2(\xi + u(k))} = 0, \quad a.e. \quad \xi \in K,$$

for  $\varphi_1, \varphi_2 \in L^2(K)$ . This follows by noting that  $V_{\varphi_1}$  is orthogonal to  $V_{\varphi_2}$  if and only if  $\langle T_k \varphi_1, \varphi_2 \rangle = 0$ , for all  $k \in \mathbb{N}_0$ . Now, using the argument of standard periodization, we obtain

$$\langle T_k \varphi_1, \varphi_2 \rangle = \int_{\mathcal{O}} \chi_{u(k)}(\xi) \sum_{p \in \mathbb{N}_0} \widehat{\varphi}_1(\xi + u(p)) \overline{\widehat{\varphi}_2(\xi + u(p))} d\xi = 0,$$

for all  $k \in \mathbb{N}_0$ , and hence by using the property of Fourier coefficients of the integral periodic function we conclude the condition.

Next theorem plays an important role to find building blocks of all  $\mathcal{Z}$ -TI spaces.

**Proposition 3.2.1. (A)** *Let  $V_\varphi$  be a  $\mathcal{Z}$ -PTI space. Then  $f \in V_\varphi$  if and only if  $\widehat{f}(\xi) = r(\xi)\widehat{\varphi}(\xi)$ , for some integral periodic function  $r \in L^2(\mathcal{O}, w)$ , where*

$$w(\xi) = \sum_{k \in \mathbb{N}_0} |\widehat{\varphi}(\xi + u(k))|^2.$$

The support of  $w$  is known as the spectrum of  $V_\varphi$  which is denoted by  $\Omega$ .

(B) Let  $\varphi \in L^2(K)$ . Then a necessary and sufficient condition for the system  $\{T_k\varphi : k \in \mathbb{N}_0\}$  to be a Parseval frame for the  $\mathcal{Z}$ -PTI space  $V_\varphi$  is as follows:

$$\sum_{k \in \mathbb{N}_0} |\widehat{\varphi}(\xi + u(k))|^2 = \chi_\Omega(\xi), \quad a.e.$$

**Proof.** (A) This follows by noting that  $V_\varphi = \overline{\mathcal{A}_\varphi}$ ,  $L^2(\mathcal{O}, w) = \overline{\mathcal{P}_\varphi}$  and the operator  $U : \mathcal{A}_\varphi \rightarrow \mathcal{P}_\varphi$  defined by  $U(f)(\xi) = r(\xi)$  is an isometry which is onto, where  $\mathcal{A}_\varphi = \text{span}\{T_k\varphi : k \in \mathbb{N}_0\}$ , and  $\mathcal{P}_\varphi$  is the space of all integral periodic trigonometric polynomials  $r$  with the  $L^2(\mathcal{O}, w)$  norm

$$\|r\|_{L^2(\mathcal{O}, w)}^2 = \int_{\mathcal{O}} |r(\xi)|^2 w(\xi) d\xi, \quad \text{where} \quad w(\xi) = \sum_{k \in \mathbb{N}_0} |\widehat{\varphi}(\xi + u(k))|^2.$$

Here,  $f \in \mathcal{A}_\varphi$  if and only if for  $r \in \mathcal{P}_\varphi$ ,  $\widehat{f}(\xi) = r(\xi)\widehat{\varphi}(\xi)$ , where  $r(\xi) = \sum_{k \in \mathbb{N}_0} a_k \overline{\chi_{u(k)}(\xi)}$ , for a finite number of non-zero elements of  $\{a_k\}_{k \in \mathbb{N}_0}$ . Now, by splitting the integral into cosets of  $\mathcal{O}$  in  $K$  and using the fact of integral periodicity of  $r$ , we have

$$\|f\|_2^2 = \int_{\mathcal{O}} \sum_{k \in \mathbb{N}_0} \left| \widehat{f}(\xi + u(k)) \right|^2 d\xi = \int_{\mathcal{O}} |r(\xi)|^2 \sum_{k \in \mathbb{N}_0} |\widehat{\varphi}(\xi + u(k))|^2 d\xi = \|r\|_{L^2(\mathcal{O}, w)}^2,$$

which shows that the operator  $U$  is an isometry.

(B) Notice that for every  $f \in V_\varphi$ , we have  $\widehat{f}(\xi) = r(\xi)\widehat{\varphi}(\xi)$ , for some integral periodic function  $r \in L^2(\mathcal{O}, w)$ , and hence

$$\sum_{k \in \mathbb{N}_0} |\langle f, T_k\varphi \rangle|^2 = \sum_{k \in \mathbb{N}_0} \left| \int_{\mathcal{O}} r(\xi) w(\xi) \chi_{u(k)}(\xi) d\xi \right|^2 = \int_{\mathcal{O}} |r(\xi)|^2 |w(\xi)|^2 d\xi.$$

Therefore, we have condition

$$\int_{\mathcal{O}} |r(\xi)|^2 |w(\xi)| d\xi = \int_{\mathcal{O}} |r(\xi)|^2 |w(\xi)|^2 d\xi,$$

since for every  $f \in V_\varphi$ , we have  $\|f\|_2^2 = \int_{\mathcal{O}} |r(\xi)|^2 |w(\xi)| d\xi$ . That means,

$$\int_{\mathcal{O}} |r(\xi)|^2 w(\xi) (\chi_\Omega(\xi) - w(\xi)) d\xi = 0,$$

holds for all integral periodic functions  $r \in L^2(\mathcal{O}, w)$  if and only if  $w(\xi) = \chi_\Omega(\xi)$ , *a.e.*  $\square$

Notice that for all  $\xi \in K$ ,  $\sum_{k \in \mathbb{N}_0} |\widehat{\varphi}(\xi + u(k))|^2 \neq 1$ , if  $\widehat{\varphi} = I_{\mathfrak{P}}$  since  $\mathfrak{P} \subset \mathcal{O}$  and  $\{\mathcal{O} + u(k) : k \in \mathbb{N}_0\}$  is a measurable partition of  $K$ . In the sequel of development of wavelets associated with an MRA on LFPC, Jiang, Li and Jin in [18] found the following result:

**Corollary 3.2.2.** *A necessary and sufficient condition to constitute an orthonormal system by  $\{T_k\varphi : k \in \mathbb{N}_0\}$  is as follows:*

$$\sum_{k \in \mathbb{N}_0} |\widehat{\varphi}(\xi + u(k))|^2 = 1, \quad \text{a.e.},$$

for any  $\varphi \in L^2(K)$ .

Next, consider a mapping  $\mathcal{T} : L^2(K) \rightarrow L^2(\mathcal{O}, l^2(\mathcal{Z}))$  defined by

$$\mathcal{T}f(\xi) = \left( \widehat{f}(\xi + u(p)) \right)_{p \in \mathbb{N}_0}$$

which is an isometric isomorphism between  $L^2(K)$  and  $L^2(\mathcal{O}, l^2(\mathcal{Z}))$  that is an easy consequence of Plancherel theorem. Following is an immediate application of above fiberization map:

**Proposition 3.2.3.** *Let  $V$  be a  $\mathcal{Z}$ -TI subspace of  $L^2(K)$ . Then the image of  $V$  under the fiberization map  $\mathcal{T}$  is given as follows:*

$$\mathcal{T}(V) = \{F \in L^2(\mathcal{O}, l^2(\mathcal{Z})) : F(\xi) \in J(\xi)\}, \text{ where } J(\xi) = \{\mathcal{T}f(\xi) : f \in V\}, \text{ for } \xi \in \mathcal{O}.$$

The mapping  $J$  from  $\mathcal{O}$  to  $\{\text{closed subspaces of } l^2(\mathcal{Z})\}$  is known as range function, and the dimension of  $J(\xi)$  is the multiplicity function of  $V$  which is denoted by  $m_V(\xi) (= \dim J(\xi))$ , for  $\xi \in \mathcal{O}$ . Throughout, we assume that the range function  $J$  is measurable.

Moreover, the multiplicity function satisfies the following condition:

$$m_V(\xi) = \sum_{n=1}^M \sum_{k \in \mathbb{N}_0} |\widehat{\varphi}_n(\xi + u(k))|^2,$$

where  $V = \oplus_{i=1}^M V_{\varphi_i}$ , and  $\varphi_i$  is a Parseval frame generator for the  $\mathcal{Z}$ -PTI space  $V_{\varphi_i}$ .

**Proof.** Let  $V$  be a  $\mathcal{Z}$ -TI subspace of  $L^2(K)$ . Then, there exists a countable set of functions  $\{\varphi_i\}_{i=1}^M$  belonging to  $V$  such that  $V = \oplus_{i=1}^M V_{\varphi_i}$ , where  $M$  is a natural number or infinity. Let us consider a  $\mathcal{Z}$ -PTI space  $V_{\varphi_i}$  with a Parseval frame generator  $\varphi_i$  and a spectrum  $\Omega_i$ . Then it is enough to find the image of  $V_{\varphi_i}$  under the transformation  $\mathcal{T}$ . For this, as every function  $f \in V_{\varphi_i}$  satisfies  $\widehat{f}(\xi) = r(\xi)\widehat{\varphi}_i(\xi)$ , for some integral periodic function  $r \in L^2(\mathcal{O}, \Omega_i)$ , we have

$$\mathcal{T}(f)(\xi) = \left( \widehat{f}(\xi + u(p)) \right)_{p \in \mathbb{N}_0} = r(\xi) (\widehat{\varphi}_i(\xi + u(p)))_{p \in \mathbb{N}_0} = r(\xi) \mathcal{T}(\varphi_i)(\xi),$$

and hence, we obtain

$$\begin{aligned} \mathcal{T}(V_{\varphi_i}) &= \{F \in L^2(\mathcal{O}, l^2(\mathcal{Z})) : F(\xi) = r(\xi) \mathcal{T}(\varphi_i)(\xi), r \in L^2(\mathcal{O}, \Omega_i)\} \\ &= \{F \in L^2(\mathcal{O}, l^2(\mathcal{Z})) : F(\xi) \in J_i(\xi)\}, \end{aligned}$$

where  $J_i(\xi) = \text{span}\{\mathcal{T}(\varphi_i)(\xi)\}$ , for  $\xi \in \mathcal{O}$ . Therefore, the result follows for  $\mathcal{Z}$ -TI space  $V$  by noting that  $J(\xi) = \overline{\text{span}}\{\mathcal{T}(\varphi_i)(\xi) : i = 1, 2, \dots, M\} = \oplus_{i=1}^M J_i(\xi)$ .

Moreover, the multiplicity function satisfies

$$m_V(\xi) = \dim J(\xi) = \sum_{i=1}^M \dim J_i(\xi) = \sum_{i=1}^M m_{V_{\varphi_i}}(\xi) = \sum_{i=1}^M \sum_{k \in \mathbb{N}_0} |\widehat{\varphi}_i(\xi + u(k))|^2,$$

for a.e.  $\xi \in K$ . This follows by the fact that the dimension of subspace  $J_i(\xi)$  of  $l^2(\mathcal{Z})$  is one for a.e.  $\xi \in \mathcal{O} \cap \Omega_i$  and zero for a.e.  $\xi \in \mathcal{O}$  outside of  $\Omega_i$ .  $\square$

In the above theorem, if we apply  $\mathcal{T}^{-1}$  on  $\mathcal{T}(V)$  and using above facts of translation-invariant spaces, we have

$$V = \{f \in L^2(K) : \mathcal{T}f(\xi) \in J(\xi), \text{ for a.e. } \xi \in \mathcal{O}\}.$$

It can be easily seen that the space  $V$  is  $\mathcal{Z}$ -TI space.

Analogous to a result of Bownik [7], we state the following:

**Proposition 3.2.4.** *If  $V$  is a  $\mathcal{Z}$ -TI space and  $N = \|m_V\|_\infty$  ( $N$  is a natural number or infinity), then there exist a set of functions  $\{\varphi_n\}_{n=1}^N$  such that  $V = \oplus_{n=1}^N V_{\varphi_n}$ , where  $\varphi_n$  is a Parseval frame generator of  $V_{\varphi_n}$ .*

Next, we define spectral function in the context of LFPC which was studied by Rzeszutnik in [21] for the case of Euclidean spaces:

**Definition 3.2.5.** Let  $V$  be a  $\mathcal{Z}$ -TI space with the range function  $J$ . Suppose  $P_{J(\xi)}$  is the orthogonal projection on  $J(\xi)$  for a.e.  $\xi \in \mathcal{O}$ . Then the *spectral function* of  $V$  is the mapping  $\sigma_V : K \rightarrow [0, 1]$  given by

$$\sigma_V(\xi + u(k)) = \|P_{J(\xi)} e_{u(k)}\|_{l^2(\mathcal{Z})}^2, \text{ for } \xi \in \mathcal{O} \text{ and } k \in \mathbb{N}_0,$$

where  $\{e_{u(k)}\}_{k \in \mathbb{N}_0}$  denotes the standard orthonormal basis of  $l^2(\mathcal{Z})$ .

We end this section by summarizing several results of the spectral function by employing above results whose LCAG version can be found in [11].

**Theorem 3.2.6.** *The spectral function satisfies the following properties:*

- (A) *Let  $V$  be a  $\mathcal{Z}$ -TI space of  $L^2(K)$  with a set of generators  $\{\varphi_n\}_{n=1}^N$  ( $N$  is natural number or infinite). If the system  $\{T_k \varphi_n : k \in \mathbb{N}_0, 1 \leq n \leq N\}$  forms a Parseval frame for the space  $V$ , then the function*

$$\sigma_V(\xi) = \sum_{n=1}^N |\widehat{\varphi}_n(\xi)|^2,$$

*defined for a.e.  $\xi \in K$ , does not depend on the choice of generators.*

(B) Let  $\nu$  denote the set of all  $\mathcal{Z}$ -TI spaces of  $L^2(K)$ . Then there exists a unique mapping  $\sigma : \nu \longrightarrow L^\infty(K)$  such that

$$\sigma_{V_\varphi}(\xi) = \begin{cases} |\hat{\varphi}(\xi)|^2 \left( \sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 \right)^{-1} & \text{for } \xi \in \text{supp } \hat{\varphi} \\ 0 & \text{otherwise,} \end{cases}$$

which is additive on orthogonal sums, i.e.,  $V = \oplus_{i=1}^\infty V_{\varphi_i}$ , implies  $\sigma_V = \sum_{n=1}^\infty \sigma_{V_n}$ .

(C) If  $V$  is a  $\mathcal{Z}$ -TI space with a decomposition  $V = \oplus_{n=1}^N V_{\varphi_n}$ , where  $\varphi_n$  is a Parseval frame generator for  $V_{\varphi_n}$  and  $N$  is a natural number or infinity then

$$\sigma_V(\xi) = \sum_{n=1}^N |\hat{\varphi}_n(\xi)|^2, \text{ a.e.}$$

(D) If  $V$  is a  $\mathcal{Z}$ -TI space, then  $D_{\mathfrak{p}}(V)$  is  $\mathfrak{p}\mathcal{Z}$ -TI space, and  $\sigma_{D_{\mathfrak{p}}(V)}(\xi) = \sigma_V(\mathfrak{p}\xi)$ , a.e.

(E) If  $V$  is a  $\mathcal{Z}$ -TI space, then  $m_V(\xi) = \sum_{k \in \mathbb{N}_0} \sigma_V(\xi + u(k))$ , a.e.

**Proof.** (A) The result follows by noting that

$$\begin{aligned} \sigma_V(\xi + u(k)) &= \|P_{J(\xi)} e_{u(k)}\|_{l^2(\mathcal{Z})}^2 = \sum_{n=1}^N | \langle P_{J(\xi)} e_{u(k)}, \mathcal{T}(\varphi_n)(\xi) \rangle_{l^2(\mathcal{Z})} |^2 \\ &= \sum_{n=1}^N | \langle e_{u(k)}, \mathcal{T}(\varphi_n)(\xi) \rangle_{l^2(\mathcal{Z})} |^2 = \sum_{n=1}^N |\hat{\varphi}_n(\xi + u(k))|^2. \end{aligned}$$

(B) If  $V_\varphi$  is a  $\mathcal{Z}$ -PTI space, then the system  $\{T_k \psi\}_{k \in \mathbb{N}_0}$  is a Parseval frame for the  $\mathcal{Z}$ -PTI space  $V_\varphi$ , where the Fourier transform of  $\psi$  is given by

$$\hat{\psi}(\xi) = \begin{cases} \hat{\varphi}(\xi) \left( \sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 \right)^{-1/2} & \text{for } \xi \in \text{supp } \hat{\varphi} \\ 0 & \text{otherwise,} \end{cases}$$

and hence,  $\{\mathcal{T}(\psi)(\xi)\}$  is a Parseval frame for the range function  $J_{V_\varphi}(\xi)$ . So the value of  $\sigma_{V_\varphi}$  follows by noting that for all  $\xi \in \mathcal{O}$  and  $k \in \mathbb{N}_0$ , we have

$$\sigma_{V_\varphi}(\xi + u(k)) = \|P_{J_{V_\varphi}(\xi)} e_{u(k)}\|^2 = | \langle e_{u(k)}, \mathcal{T}(\psi)(\xi) \rangle |^2 = |\hat{\psi}(\xi + u(k))|^2.$$

The rest portion of the result follows by considering  $V = \oplus_{n=1}^\infty V_{\varphi_n}$ , with corresponding range functions  $J_V$  and  $J_{V_{\varphi_n}}$ , and noting that

$$\sigma_V(\xi + u(k)) = \|P_{J_V(\xi)} e_{u(k)}\|_{l^2(\mathcal{Z})}^2 = \sum_{n=1}^\infty \|P_{J_{V_{\varphi_n}}(\xi)} e_{u(k)}\|^2 = \sum_{n=1}^\infty \sigma_{V_{\varphi_n}}(\xi + u(k)).$$

(D) Consider for any  $f \in D_{\mathfrak{p}}(V)$  and  $\gamma \in \mathfrak{p}\mathcal{Z}$ , i.e.,  $f = D_{\mathfrak{p}}g$  for some  $g \in V$  and  $\mathfrak{p}^{-1}\gamma = u(l)$ , for  $u(l) \in \mathcal{Z}$ . Then for  $\xi \in K$ , we have

$$f(x - \gamma) = f(x - \mathfrak{p}u(l)) = D_{\mathfrak{p}}g(x - \mathfrak{p}u(l)) = q^{1/2}g(\mathfrak{p}^{-1}x - u(l)) = D_{\mathfrak{p}}(T_l g)(x).$$

Since  $T_l g \in V$ , this shows that  $D_{\mathfrak{p}}(V)$  is  $\mathfrak{p}\mathcal{Z}$ -TI. Next, it suffices to show the result for  $\mathcal{Z}$ -PTI space  $V_{\varphi}$  which has a Parseval frame generator  $\varphi$ . For this, let  $f \in D_{\mathfrak{p}}(V_{\varphi})$ . Then, we have

$$\begin{aligned} \|f\|_2^2 &= \|(D_{\mathfrak{p}})^{-1}f\|^2 = \sum_{k \in \mathbb{N}_0} |\langle f, D_{\mathfrak{p}}T_k \varphi \rangle|^2 = \sum_{k \in \mathbb{N}_0} \sum_{i=0}^{q-1} |\langle f, D_{\mathfrak{p}}T_{qk+i} \varphi \rangle|^2 \\ &= \sum_{k \in \mathbb{N}_0} \sum_{i=0}^{q-1} |\langle f, D_{\mathfrak{p}}T_{qk}(T_i \varphi) \rangle|^2 = \sum_{k \in \mathbb{N}_0} \sum_{i=0}^{q-1} |\langle f, T_k(D_{\mathfrak{p}}(T_i \varphi)) \rangle|^2, \end{aligned}$$

in view of the following facts: the map  $D_{\mathfrak{p}}$  is unitary on  $L^2(K)$ ; for  $k \geq 0$  and  $0 \leq i \leq q-1$ ,  $u(qk+i) = \mathfrak{p}^{-1}u(k) + u(i)$ ; and  $D_{\mathfrak{p}}T_{qk} = T_k D_{\mathfrak{p}}$ , for  $k \geq 0$ . Since  $T_i \varphi \in V_{\varphi}$ , this shows that the system  $\{D_{\mathfrak{p}}(T_i \varphi)\}_{i=0}^{q-1}$  is a Parseval frame for the space  $D_{\mathfrak{p}}(V_{\varphi})$ .

Note that (C) and (E) follow immediately.  $\square$

### 3.3. Semi-orthogonal Parseval wavelets and GMRA's

Recall that every semi-orthogonal framelet  $\Psi$  comes from a GMRA. Now we look into its converse, that is, when a GMRA gives rise to a wavelet, or a semi-orthogonal framelet with the help of knowledge of translation invariant spaces for LFPC obtained from previous subsection.

Following is a main theorem of this section:

**Theorem 3.3.1.** *Suppose that  $\Psi$  is a semi-orthogonal Parseval multiwavelet with  $L$  generators and  $V$  is the space of negative dilates of  $\Psi$ . Then,  $\{D_{\mathfrak{p}}^j(V)\}_{j \in \mathbb{Z}}$  is a GMRA such that  $m_V(\xi) < \infty$  for a.e.  $\xi$ , and*

$$\sum_{d=0}^{q-1} m_V(\mathfrak{p}(\xi + u(d))) - m_V(\xi) \leq L, \quad \text{for a.e. } \xi.$$

*Conversely, if  $\{D_{\mathfrak{p}}^j(V)\}_{j \in \mathbb{Z}}$  is a GMRA satisfying above conditions, then there exists a semi-orthogonal Parseval (multi)wavelet  $\Psi$  (with at most  $L$  generators) associated with this GMRA.*

Now, we proceed as follows to find a proof of above theorem:

**Proposition 3.3.2.** *If  $\Psi = \{\psi_l\}_{l=1}^L \subset L^2(K)$  is a semi-orthogonal Parseval multiwavelet with  $L$  generators and  $V = \bigoplus_{j < 0} D_{\mathfrak{p}}^j(W)$ , where  $W = \overline{\text{span}}\{\psi_l(\cdot - u(k)) : k \in \mathbb{N}_0, l = 1, 2, \dots, L\}$ , then*

$$m_V(\xi) = \sum_{l=1}^L \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} |\widehat{\psi}_l(\mathfrak{p}^{-j}(\xi + u(k)))|^2, \quad \text{a.e.,}$$

and hence,  $\int_{\mathcal{O}} m_V(\xi) \leq \frac{L}{q-1}$ . Moreover, it satisfies the consistency equation

$$m_V(\xi) + L \geq \sum_{d=0}^{q-1} m_V(\mathfrak{p}(\xi + u(d))), \text{ for } \xi \in K.$$

**Proof.** Since  $L^2(K) = V \oplus \oplus_{j \geq 0} D_{\mathfrak{p}}^j(W)$ , we have  $\sigma_V + \sum_{j \geq 0} \sigma_{D_{\mathfrak{p}}^j(W)} = 1$  in view of Theorem

3.2.6, and hence,  $\sigma_V(\xi) = \sum_{l=1}^L \sum_{j=1}^{\infty} |\hat{\psi}_l(\mathfrak{p}^{-j}\xi)|^2$ . This follows by noting that

$$\sigma_V(\xi) = 1 - \sum_{j \geq 0} \sigma_{D_{\mathfrak{p}}^j(W)}(\xi) = \sum_{l=1}^L \sum_{j \in \mathbb{Z}} |\hat{\psi}_l(\mathfrak{p}^j \xi)|^2 - \sum_{l=1}^L \sum_{j \geq 0} |\hat{\psi}_l(\mathfrak{p}^j \xi)|^2 = \sum_{l=1}^L \sum_{j=1}^{\infty} |\hat{\psi}_l(\mathfrak{p}^{-j} \xi)|^2,$$

because  $\sigma_{D_{\mathfrak{p}}^j(W)}(\xi) = \sum_{l=1}^L |\hat{\psi}_l(\mathfrak{p}^j \xi)|^2$ , for  $j \geq 0$ . Therefore, the first result follows by writing

$m_V(\xi) = \sum_{k \in \mathbb{N}_0} \sigma_V(\xi + u(k))$  from Theorem 3.2.6. Now, we have

$$\begin{aligned} \int_{\mathcal{O}} m_V(\xi) d\xi &= \int_{\mathcal{O}} \sum_{l=1}^L \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} |\hat{\psi}_l(\mathfrak{p}^{-j}(\xi + u(k)))|^2 d\xi = \sum_{l=1}^L \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} \int_{\mathcal{O}+u(k)} |\hat{\psi}_l(\mathfrak{p}^{-j}\xi)|^2 d\xi \\ &= \sum_{l=1}^L \sum_{j=1}^{\infty} \frac{1}{q^j} \int_K |\hat{\psi}_l(\xi)|^2 d\xi \leq \frac{L}{q-1}. \end{aligned}$$

For the consistency equation, we have

$$\begin{aligned} \sum_{d=0}^{q-1} m_V(\mathfrak{p}(\xi + u(d))) &= \sum_{d=0}^{q-1} \sum_{l=1}^L \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} |\hat{\psi}_l(\mathfrak{p}^{-j}(\mathfrak{p}\xi + \mathfrak{p}u(d) + u(k)))|^2 \\ &= \sum_{d=0}^{q-1} \sum_{l=1}^L \sum_{j=0}^{\infty} \sum_{k \in \mathbb{N}_0} |\hat{\psi}_l(\mathfrak{p}^{-j}(\xi + u(d) + \mathfrak{p}^{-1}u(k)))|^2 \\ &= \sum_{l=1}^L \sum_{k \in \mathbb{N}_0} |\hat{\psi}_l(\xi + u(k))|^2 + \sum_{l=1}^L \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} |\hat{\psi}_l(\mathfrak{p}^{-j}(\xi + u(k)))|^2 \\ &\leq L + m_V(\xi). \end{aligned} \quad \square$$

**Corollary 3.3.3.** *If  $f \in L^2(K)$ , then the collection  $\{f(\cdot - u(k)) : k \in \mathbb{N}_0\}$  is a Parseval sequence if and only if  $m_f$  satisfies the following consistency equation*

$$\sum_{d=0}^{q-1} m_f(\mathfrak{p}(\xi + u(d))) \leq 1 + m_f(\xi),$$

where  $m_f(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} |\hat{f}(\mathfrak{p}^{-j}(\xi + u(k)))|^2$ .

**Corollary 3.3.4.** *If  $f \in L^2(K)$ , then the collection  $\{f(\cdot - u(k)) : k \in \mathbb{N}_0\}$  is an orthonormal system if and only if  $m_f$  satisfies the following consistency equation*

$$\sum_{d=0}^{q-1} m_f(\mathfrak{p}(\xi + u(d))) = 1 + m_f(\xi),$$

$$\text{where } m_f(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} |\widehat{f}(\mathfrak{p}^{-j}(\xi + u(k)))|^2.$$

Now, the following result gives a characterization of a multiwavelet associated with an MRA:

**Theorem 3.3.5.** *If  $\Psi = \{\psi_1, \psi_2, \dots, \psi_{q-1}\} \subset L^2(\mathbb{R})$  is an orthonormal multiwavelet, and  $m$  is its associated multiplicity function, then  $\Psi$  is associated with an MRA if and only if  $m \equiv 1$ , a.e.*

**Proof.** Suppose  $\Psi$  is an MRA multiwavelet. Then there exists  $\varphi \in V$  such that the system  $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$  is an orthonormal basis for  $V$ , and hence,  $\sum_{k \in \mathbb{N}_0} |\widehat{\varphi}(\xi + u(k))|^2 = 1$ , a.e. Therefore, we have  $m(\xi) = 1$  since  $V = \overline{\text{span}}\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$  is a  $\mathcal{Z}$ -PTI space. Conversely, assume that  $\Psi \in L^2(\mathbb{R})$  is a multiwavelet, and  $m$  is its associated multiplicity function such that  $m(\xi) = 1$ . Then, we have to show that  $\Psi$  is an MRA multiwavelet. For this, consider  $V = \oplus_{j < 0} D_{\mathfrak{p}}^j(W)$ , where  $W = \overline{\text{span}}\{\psi_l(\cdot - u(k)) : k \in \mathbb{N}_0, l = 1, 2, \dots, q-1\}$ . Then, the multiplicity function  $m(\xi) = 1$ , and hence,  $V = V_{\varphi}$ , where  $\varphi$  is a Parseval frame generator for  $V_{\varphi}$  and spectrum of  $V_{\varphi}$  is equal to  $K$  in view of Proposition 3.2.4. Therefore, we have  $\sum_{k \in \mathbb{N}_0} |\widehat{\varphi}(\xi + u(k))|^2 = 1$ , a.e. and hence,  $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$  is an orthonormal basis for  $V$ . □

Next, we define multiplicity function associated with a wavelet as follows:

**Definition 3.3.6.** If  $\Psi$  is a multiwavelet, then there exists a multiplicity function associated to it. This function is called the *wavelet multiplicity function*.

Let  $\Psi = \{\psi_l\}_{l=1}^L$  be a multiwavelet on  $L^2(K)$ , and consider the wavelet dimension function  $D_{\Psi}(\xi)$  for LFPC (defined in [5]) which is given by

$$D_{\Psi}(\xi) = \sum_{l=1}^L \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} |\widehat{\psi}_l(\mathfrak{p}^{-j}(\xi + u(k)))|^2.$$

**Corollary 3.3.7.** *Let  $\Psi$  be a multiwavelet, and let  $m : \mathcal{O} \rightarrow \mathbb{N}_0$  be its associated multiplicity function. Then  $m(\xi) = D_{\Psi}(\xi)$ .*

**Proof of Theorem 3.3.1.** Suppose that  $\Psi$  is a semi-orthogonal Parseval multiwavelet with  $L$  generators and the spaces  $V$  and  $W$  are defined as follows:

$$W = \overline{\text{span}}\{T_k \psi : k \in \mathbb{N}_0, \psi \in \Psi\}, \text{ and } V = \overline{\text{span}}\{\psi_{j,k} : j < 0, k \in \mathbb{N}_0, \psi \in \Psi\}.$$



Then, we have

$$\begin{aligned} \int_{\mathcal{O}} m_V(\xi) d\xi &= \int_K \sigma_V(\xi) d\xi = \sum_{\psi \in \Psi} \sum_{j=1}^{\infty} \int_K |\widehat{\psi}(\mathfrak{p}^{-j}\xi)|^2 = \sum_{\psi \in \Psi} \|\psi\|^2 / (q-1) \\ &\leq L/(q-1) < \infty. \end{aligned}$$

Hence,  $m_V(\xi) < \infty$ . Since  $W \oplus V = D_{\mathfrak{p}}(V)$ , We have

$$\sigma_W(\xi) + \sigma_V(\xi) = \sigma_{D_{\mathfrak{p}}(V)}(\xi) = \sigma_V(\mathfrak{p}\xi).$$

This implies that

$$m_W(\xi) + m_V(\xi) = \sum_{d=0}^{q-1} m_V(\mathfrak{p}(\xi + u(d))).$$

Since  $m_W(\xi) \leq L$ , we get the result.

Conversely, from

$$\sum_{d=0}^{q-1} m_V(\mathfrak{p}(\xi + u(d))) - m_V(\xi) \leq L$$

and

$$m_W(\xi) + m_V(\xi) = \sum_{d=0}^{q-1} m_V(\mathfrak{p}(\xi + u(d))),$$

we have  $m_W(\xi) \leq L$ . By Theorem 3.2.4, this implies that  $W$  has a set  $\Psi$  having generators less than or equal to  $L$ . Since  $V = \bigoplus_{j \leq -1} D_{\mathfrak{p}}^j(W)$ , we infer that  $\Psi$  is a semi-orthogonal Parseval multiwavelet associated with the GMRA  $\{D_{\mathfrak{p}}^j(V)\}_{j \in \mathbb{Z}}$ .  $\square$

**Corollary 3.3.8.** *Let  $V$  be a  $\mathcal{Z}$ -TI space such that the multiplicity function of  $V$  is integrable and satisfies the following consistency equation*

$$m_V(\xi) + L = \sum_{d=0}^{q-1} m_V(\mathfrak{p}(\xi + u(d))),$$

*then there exists a set of functions  $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\}$  in  $D_{\mathfrak{p}}(V) \ominus V \equiv W$  such that the system  $\{\psi_l(\cdot - u(k)) : k \in \mathbb{N}_0, 1 \leq l \leq L\}$  is an orthonormal basis for  $W$ .*

#### 4. Bandlimited Wavelets for LFPC

The present section is devoted to the study of characterizations of bandlimited Parseval multiwavelets as well as Parseval scaling functions for LFPC.

**Proposition 4.1.** *Let  $\Psi = \{\psi_m\}_{m=1}^L \subset L^2(K)$  be such that for each  $m \in \{1, 2, \dots, L\}$ ,  $|\widehat{\psi}_m| = \chi_{W_m}$ , and  $W = \bigcup_{m=1}^L W_m$  is a union of measurable subsets of  $K$ . Then  $\Psi$  is a Parseval multiwavelet in  $L^2(K)$  if and only if the following hold:*

- (i)  $\{\mathfrak{p}^j W : j \in \mathbb{Z}\}$  is a measurable partition of  $K$ , and
- (ii) for each  $m \in \{1, 2, \dots, L\}$ , the set  $\{W_m + u(k) : k \in \mathbb{N}_0\}$  is a measurable partition of a subset of  $K$ .

In this case, for  $j \in \mathbb{Z}$ ,  $|\mathfrak{p}^j W_m \cap W_{m'}| = 0$ , where  $m, m' \in \{1, 2, \dots, L\}$ , and  $m \neq m'$ . The set  $W$  is known as Parseval multiwavelet set of order  $L$  in  $K$ .

**Proof.** Let  $\Psi = \{\psi_m\}_{m=1}^L \subset L^2(K)$  be such that  $|\widehat{\psi}_m| = \chi_{W_m}$ , where  $W = \bigcup_{m=1}^L W_m$  is a measurable subset of  $K$ . Then, the condition (i) of Theorem 3.1.2 yields that  $\bigcup_{j \in \mathbb{Z}} \mathfrak{p}^j W = K$ , a.e., that is equivalent to (i), which also gives that for  $j \geq 0$ ,  $|\mathfrak{p}^j W_m \cap W_{m'}| = 0$ , for each  $m, m' \in \{1, 2, \dots, L\}$ , and  $m \neq m'$ . Further in view of Proposition 3.2.1 and for each  $m \in \{1, 2, \dots, L\}$ , the system  $\{\psi_m(\cdot - u(k)) : k \in \mathbb{N}_0\}$ , is a Parseval frame for  $\overline{\text{span}}\{\psi_m(\cdot - u(k)) : k \in \mathbb{N}_0\}$  in  $L^2(K)$  if and only if  $\sum_{k \in \mathbb{N}_0} \left| \widehat{\psi}_m(\xi + u(k)) \right|^2 = \sum_{k \in \mathbb{N}_0} \chi_{W_m}(\xi + u(k)) \leq 1$ , a.e., that is equivalent to the (ii). In this case  $\{f \in L^2(K) : \text{supp } \widehat{f} \subset W\} = \overline{\text{span}}\{\psi(\cdot - u(k)) : \psi \in \Psi, k \in \mathbb{N}_0\} =: W_0$ . By scaling  $W_0$  for any  $j \in \mathbb{Z}$ , we have  $D_{\mathfrak{p}}^j W_0 = \overline{\text{span}}\{D_{\mathfrak{p}}^j \psi(\cdot - u(k)) : \psi \in \Psi, k \in \mathbb{N}_0\} = \{f \in L^2(K) : \text{supp } \widehat{f} \subset \mathfrak{p}^{-j} W\}$ . Therefore,  $\Psi$  is a Parseval multiwavelet in  $L^2(K)$  if and only if the system  $\mathcal{A}(\Psi)$  forms a Parseval frame for  $L^2(K)$  if and only if  $\bigoplus_{j \in \mathbb{Z}} D_{\mathfrak{p}}^j W_0 = L^2(K)$  and (ii) hold, which is true if and only if (i) and (ii) hold.  $\square$

**Corollary 4.2.** Let  $\Psi = \{\psi_m\}_{m=1}^L \subset L^2(K)$  be such that for each  $m \in \{1, 2, \dots, L\}$ ,  $|\widehat{\psi}_m| = \chi_{W_m}$ , and  $W = \bigcup_{m=1}^L W_m$  is a union of measurable subsets of  $K$ . Then  $\Psi$  is a multiwavelet in  $L^2(K)$  if and only if the following hold:

- (i)  $\{\mathfrak{p}^j W : j \in \mathbb{Z}\}$  is a measurable partition of  $K$ , and
- (ii) for each  $m \in \{1, 2, \dots, L\}$ , the system  $\{W_m + u(k) : k \in \mathbb{N}_0\}$  is a measurable partition of  $K$ .

In this case, for  $j \in \mathbb{Z}$ ,  $|\mathfrak{p}^j W_m \cap W_{m'}| = 0$ , where  $m, m' \in \{1, 2, \dots, L\}$ , and  $m \neq m'$ . The set  $W$  is known as multiwavelet set of order  $L$  in  $K$ .

The result given below is a necessary and sufficient conditions of Parseval scaling functions for LFPC:

**Proposition 4.3.** Let  $\varphi \in L^2(K)$ . Then,  $\varphi$  is a Parseval scaling function associated to a PMRA if and only if the following conditions hold:

- (i) the system  $\{\varphi(\cdot - u(k))\}_{k \in \mathbb{N}_0}$  is a Parseval frame for  $\overline{\text{span}}\{\varphi(\cdot - u(k))\}_{k \in \mathbb{N}_0}$  in  $L^2(K)$ ,
- (ii)  $\lim_{j \rightarrow \infty} |\widehat{\varphi}(\mathfrak{p}^j \xi)| = 1$ , a.e.  $\xi \in K$ , and
- (iii) there exists an integral periodic function  $m_0$  in  $L^2(\mathcal{O})$  such  $\widehat{\varphi}(\xi) = m_0(\xi) \widehat{\varphi}(\mathfrak{p}\xi)$ , a.e.  $\xi \in K$ .

**Proof.** Those (ii) and (iii) are straightforward in view of Theorem 5.1 of [4], and (i) follows by noting Proposition 3.2.1.  $\square$

Next, we illustrate a characterization of a Parseval scaling function  $\varphi$  such that  $|\widehat{\varphi}| = \chi_S$ , for some measurable set  $S$  of  $K$ . Such set  $S$  is known as *Parseval scaling set*.

**Proposition 4.4.** *A function  $\varphi$  such that  $|\widehat{\varphi}| = \chi_S$ , for some measurable set  $S$  of  $K$  is a Parseval scaling function of a PMRA if and only if*

- (i)  $\{S + 2k\pi : k \in \mathbb{Z}^n\}$  is a measurable partition of a subset of  $K$ ,
- (ii)  $\bigcup_{j \in \mathbb{Z}} \mathbf{p}^{-j}S = K$ , and
- (iii)  $S \subset \mathbf{p}^{-1}S$ .

Moreover, Parseval multiwavelet set(s)  $W$  associated to PMRA can be obtained by  $W = \mathbf{p}^{-1}S \setminus S$ , and hence  $S = \bigcup_{j \in \mathbb{N}} \mathbf{p}^j W$ .

**Proof.** Suppose  $\varphi$  is a Parseval scaling function such that  $|\widehat{\varphi}| = \chi_S$ . Then from (i) of Proposition 4.3 and Proposition 3.2.1, we have  $\sum_{k \in \mathbb{N}} |\widehat{\varphi}(\xi + u(k))|^2 = \chi_\Omega(\xi) \Rightarrow \sum_{k \in \mathbb{N}} \chi_S(\xi + u(k)) = \chi_\Omega(\xi) \Rightarrow (\{S + u(k) : k \in \mathbb{N}\}, \text{ a measurable partition of a subset of } K)$ . From (iii) of Proposition 4.3,  $\widehat{\varphi}(\xi) = m(\xi)\widehat{\varphi}(\mathbf{p}\xi) \Rightarrow |\widehat{\varphi}(\xi)| = |m(\xi)||\widehat{\varphi}(\mathbf{p}\xi)| \Rightarrow (\chi_S \leq \chi_{\mathbf{p}^{-1}S}, \text{ since } |m(\xi)| \leq 1) \Rightarrow S \subset \mathbf{p}^{-1}S$ ; and from (ii) of Proposition 4.3,  $\lim_{j \rightarrow +\infty} |\widehat{\varphi}(\mathbf{p}^j \xi)|^2 = 1 \Rightarrow \lim_{j \rightarrow +\infty} \chi_S(\mathbf{p}^j \xi) = 1 \Rightarrow (\text{for every } \epsilon > 0, \text{ there is an } N \in \mathbb{N} \text{ such that } |\chi_S(\mathbf{p}^j \xi) - 1| = |\chi_S(\mathbf{p}^j \xi) - \chi_K(\xi)| = |\chi_{(K \setminus \mathbf{p}^{-j}S)}| < \epsilon, \text{ whenever } j > N) \Rightarrow \bigcup_{j \in \mathbb{Z}} \mathbf{p}^{-j}S = K$ , since  $S \subset \mathbf{p}^{-1}S$ .

Conversely, suppose  $\{S + u(k) : k \in \mathbb{N}\}$  is a measurable partition of a subset of  $K$ ,  $\bigcup_{j \in \mathbb{Z}} \mathbf{p}^{-j}S = K$ , and  $S \subset \mathbf{p}^{-1}S$ . Then to prove that  $\varphi$  such that  $|\widehat{\varphi}| = \chi_S$  is a Parseval scaling function, it suffices to see the conditions (i), (ii) and (iii). The measurable partition  $\{S + u(k) : k \in \mathbb{N}\}$ , of a subset of  $K$  implies  $\sum_{k \in \mathbb{N}} \chi_S(\xi + u(k)) = \chi_\Omega(\xi) \Rightarrow \sum_{k \in \mathbb{N}} |\widehat{\varphi}(\xi + u(k))|^2 = \chi_\Omega(\xi)$ ;  $S \subset \mathbf{p}^{-1}S \Rightarrow \chi_S \leq \chi_{\mathbf{p}^{-1}S} \Rightarrow |\widehat{\varphi}(\xi)| = |m(\xi)||\widehat{\varphi}(\mathbf{p}\xi)|$ , where  $|m(\xi)| = \sum_{k \in \mathbb{N}} \chi_{\mathbf{p}S}(\xi + 2k\pi) \Rightarrow \widehat{\varphi}(\xi) = m(\xi)\widehat{\varphi}(\mathbf{p}\xi)$ , where  $m(\xi) = \theta(\xi) \sum_{k \in \mathbb{N}} \chi_{\mathbf{p}S}(\xi + u(k))$ , for some unimodular, integral periodic function  $\theta$ ; and  $\bigcup_{j \in \mathbb{N}} \mathbf{p}^{-j}S = K \Rightarrow \text{for } j > N, \bigcup_{j > N} \mathbf{p}^{-j}S = K$ , since  $S \subset \mathbf{p}^{-1}S \Rightarrow \lim_{j \rightarrow +\infty} \chi_S(\mathbf{p}^j \xi) = 1 \Rightarrow \lim_{j \rightarrow +\infty} |\widehat{\varphi}(\mathbf{p}^j \xi)|^2 = 1$ .  $\square$

**Corollary 4.5.** *A function  $\varphi$  such that  $|\widehat{\varphi}| = \chi_S$ , for some measurable set  $S$  of  $K$  is a orthonormal scaling function of an MRA if and only if*

- (i)  $\{S + 2k\pi : k \in \mathbb{Z}^n\}$  is a measurable partition of  $K$ ,
- (ii)  $\bigcup_{j \in \mathbb{Z}} \mathbf{p}^{-j}S = K$ , and
- (iii)  $S \subset \mathbf{p}^{-1}S$ .

Moreover, multiwavelet set(s)  $W$  of order  $q - 1$  associated to MRA can be obtained by  $W = \mathbf{p}^{-1}S \setminus S$ , and hence  $S = \bigcup_{j \in \mathbb{N}} \mathbf{p}^j W$ , call as orthonormal scaling set.

Next, we provide some examples of orthonormal and Parseval scaling sets.

**Example 4.6. (A).** Let us consider Example 3.1.4 in which the multiwavelet set  $W$  of order  $q - 1$  is

$$W = \bigcup_{j=1}^{q-1} (\mathcal{O} + u(j)) = \mathfrak{P}^{-1} \setminus \mathcal{O}.$$

Further, we consider the set  $S = \bigcup_{j \in \mathbb{N}} \mathfrak{p}^j W = \bigcup_{j \in \mathbb{N}} \mathfrak{p}^j (\mathfrak{P}^{-1} \setminus \mathcal{O})$ . Then, we have  $S = \mathcal{O}$  since  $\mathfrak{p}\mathcal{O} \subset \mathcal{O}$  and  $\mathfrak{P}^{-1} = \mathfrak{p}^{-1}\mathcal{O}$ , and the set  $S = \mathcal{O}$  satisfies conditions (i), (ii) and (iii) of Corollary 4.5. Therefore  $S$  is an orthonormal scaling set in the local field  $K$  of positive characteristic, and hence  $\Psi$  defined in Example 3.1.4 is associated with an MRA whose scaling function  $\varphi$  is defined by  $\widehat{\varphi} = \chi_{\mathcal{O}}$ .

**(B).** Let us consider Example 3.1.5 (a) in which the Parseval multiwavelet set  $W$  of order 1 is  $W = \mathfrak{p}^m \mathcal{O} \setminus \mathfrak{p}^{m+1} \mathcal{O}$ , where  $m \in \mathbb{N}$ . Further, we consider the set  $S = \bigcup_{j \in \mathbb{N}} \mathfrak{p}^j W = \mathfrak{p}^{m+1} \mathcal{O}$ .

Then,  $S$  is a Parseval scaling set in view of conditions (i), (ii) and (iii) of Proposition 4.4, and hence  $\psi$  defined in Example 3.1.5 (a) is associated with a PMRA whose Parseval scaling function  $\varphi$  is defined by  $\widehat{\varphi} = \chi_{\mathfrak{p}^{m+1} \mathcal{O}}$ .

**(C).** Let us consider Example 3.1.5 (b) in which the Parseval multiwavelet set  $W$  of order  $q - 1$  is  $W = \bigcup_{j=1}^{q-1} \mathfrak{p}^m (\mathcal{O} + u(j)) = \mathfrak{p}^m (\mathfrak{P}^{-1} \setminus \mathcal{O})$ , where  $m \in \mathbb{N}$ . Further, we consider the set  $S = \bigcup_{j \in \mathbb{N}} \mathfrak{p}^j W = \mathfrak{p}^m \mathcal{O}$ . Then,  $S$  is a Parseval scaling set in view of conditions (i), (ii) and (iii) of Proposition 4.4, and hence  $\Psi$  defined in Example 3.1.5 (b) is associated with a PMRA whose Parseval scaling function  $\varphi$  is defined by  $\widehat{\varphi} = \chi_{\mathfrak{p}^m \mathcal{O}}$ .

## References

- [1] S. Albeverio, S. Evdokimov and M. Skopina,  $p$ -adic multiresolution analysis and wavelet frames, J. Fourier Anal. Appl. 16(2010), 693-714.
- [2] L.W. Baggett, H.A. Medina and K.D. Merrill, Generalized multi-resolution analyses and a construction procedure for all wavelet sets in  $\mathbb{R}^n$ , J. Fourier Anal. Appl. 5(6) (1999), 563-573.
- [3] D. Bakić, Semi-orthogonal Parseval frame wavelets and generalized multiresolution analyses, Appl. Comput. Harmon. Anal. 21(3) (2006), 281-304.
- [4] B. Behera and Q. Jahan, Multiresolution analysis on local fields and characterization of scaling functions, Adv. Pure Appl. Math. 3(2) (2012), 181-202.
- [5] B. Behera, and Q. Jahan, Characterization of wavelets and MRA wavelets on local fields of positive characteristic, Collect. Math. 66(1) (2015), 33-53.
- [6] J.J. Benedetto and R.L. Benedetto, A wavelet theory for local fields and related groups, J. Geom. Anal. 14(3) (2004), 423-456.

- [7] M. Bownik, The structure of shift-invariant subspaces of  $L^2(\mathbb{R}^n)$ , J. Funct. Anal. 177(2) (2000), 282-309.
- [8] M. Bownik, Z. Rzeszotnik and D. Speegle, A characterization of dimension functions of wavelets, Appl. Comput. Harmon. Anal. 10(1) (2001), 71-92.
- [9] M. Bownik and Z. Rzeszotnik, On the existence of multiresolution analysis of framelets, Math. Ann. 332(4) (2005), 705-720.
- [10] M. Bownik, Baggett's problem for frame wavelets, Representations, wavelets, and frames, 153-173, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2008.
- [11] M. Bownik and K.A. Ross, The structure of translation-invariant spaces on locally compact abelian groups, J. Fourier Anal. Appl., online.
- [12] C.K. Chui and W. He, Compactly supported tight frames associated with refinable functions, Appl. Comput. Harmon. Anal. 8(3)(2000), 293-319.
- [13] B. Currey and A. Mayeli, Gabor fields and wavelet sets for the Heisenberg group, Monatsh. Math. 162(2) (2011), 119-142.
- [14] B. Currey, A. Mayeli and V. Oussa, Characterization of shift-invariant spaces on a class of nilpotent Lie groups with applications, J. Fourier Anal. Appl. 20(2)(2014), 384-400.
- [15] D.E. Dutkay, Some equations relating multiwavelets and multiscaling functions, J. Funct. Anal. 226(1) (2005), 1-20.
- [16] Yu A. Farkov, Orthogonal wavelets on locally compact abelian groups, Funct. Anal. Appl. 31(4)(1997), 294-296.
- [17] Yu A. Farkov, Multiresolution analysis and wavelets on Vilenkin groups, Facta Universitatis (NIS) Ser. Electron. Energ. 21(2008), 309-325.
- [18] H.K. Jiang, D.F. Li and N. Jin, Multiresolution analysis on local fields, J. Math. Anal. Appl., 294(2)(2004), 523-532.
- [19] W.C. Lang, Wavelet analysis on the Cantor dyadic group, Houst. J. Math. 24(3)(1998), 533-544.
- [20] D.F. Li and H.K. Jiang, The necessary condition and sufficient condition for wavelet frame on local fields, J. Math. Anal. Appl. 345(1)(2008), 500-510.
- [21] Z. Rzeszotnik, Characterization theorems in the theory of wavelets, Ph.D. thesis, Washington University, 2000.
- [22] F.A. Shah, Frame multiresolution analysis on local fields of positive characteristic, J. Oper. 2015, Art. ID 216060, 8 pp.
- [23] F.A. Shah and L. Debnath, Tight wavelet frames on local fields, Analysis (Berlin) 33(3) (2013), 293-307.
- [24] N.K. Shukla and A. Vyas, Multiresolution analysis through low-pass filter on local fields of positive characteristic, Complex Anal. Oper. Theory 9(3) (2015), 631-652.
- [25] M. H. Taibleson, Fourier analysis on local fields, Princeton Univ. Press, Princeton, N.J., 1975.